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The Model of Real Data Constructing Using Fractional Brownian Motion

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Abstract

In this paper we investigate the properties of the fractional Brownian motion as a basic process of stochastic time series models. New method of estimating the Hurst exponent is substantiated. Stochastic model, which is representing a time series analysis in the form of increments converted to fractional Brownian motion. The method of checking the adequacy of the proposed models. The research results are implemented in software for the simulation and analysis temporal data.

Keywords: Stochastic model; Fractional Brownian motion; Estimation of parameters

Introduction

Let $x(t), t \in [0,T]$ is an observed trajectory, which is describing the stochastic evolution of some dynamic object. Mathematical model of this trajectory is defined as a random process, $\xi(t)$. Where: $x(t) = X(t)$, $X(\cdot)$ is realization of the process ξ.

As a rule, we chose as a model random process with known characteristics. Direct use of this definition requires broad classes of these processes. On the other hand, this class includes Gauss and Markov processes. Let's introduce another definition of continuous mathematical models for the observed trajectory $x(\cdot) \in C(0;T)$ using nonlinear conversion.

Definition

Mathematical model of observed trajectory x (t) is a pair (Φ, ξ), where $x(t) = \Phi(X(\cdot))(t)$, $\xi(t)$ is a random process with known characteristics, Φ is a reversible conversion in $c\,(0, t)$.

Let's assume $t = t_k, k = 1,..,n,\ t_k = t_1 + (k-1)\frac{T-0}{n}, t_1 = 0,\ x_k = \Phi(X(\cdot))(t_k)$ is a model of observed time series $x_1 ... x_n$.

Let's call ξ as a basic process of the model. Levy processes with independent stationary increments have been considered as basic for models of time series (particularly financial) [1-4]. The next step in the development of the models is transition to diffusion processes. For example, diffusion model of stock price $S(t)$ is obtained from the following considerations:

$$S(t) = S(0)\exp\{\sigma w(t) + \mu t\},\ X(t) = \ln S(t),$$

Where $w(t)$ is a standard Wiener process, σ is volatility and interest rate, μ is a constant. Then let's propose the equation:

$$dS(t) = \sigma S(t)\, dw(t) + \mu S(t)\, dt$$

Which can be interpreted as a stochastic equation Ito and its solution could be written as a geometric (economic) Brownian motion:

$$S(t) = S(0)\exp\left\{\sigma\, w(t) + \left(\mu - \frac{\sigma^2}{2}\right)t\right\} \quad (1)$$

For the model (1) of a stock price have been obtained a number of known results, including the Black-Scholes formula for a rational option pricing [5-8]. The main drawback of Levy processes (and diffusion) is their Markov. Thus, the Markov property:

$$P\{X(t_n) \in \Delta | X(t_1) = a_1,.., X(t_{n-1}) = a_{n-1}\} =$$

$$= P(X(t_n) \in \Delta | X(t_{n-1}) = a_{n-1})$$

A priori satisfies only the simplest physical phenomena. The absence of impact on processes in biology, economics, climate, etc. looks unconvincing. In this paper we propose a non-Markovian model of the time series.

Selection of the Base Process and its Properties

One of the most popular Markov models of time series is Gaussian random process, and fractional Brownian motion [9-11]. The demand of this process is caused by "convenient" properties, which are described below.

Fractional Brownian motion is defined as a Gaussian random process with characteristics:

$$B_H(t),\ \mathrm{E}\, B_H(t) = 0,\quad B_H(0) = 0,\ \mathrm{E}\, B_H(t)\, B_H(s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

Note that with $H = 0{,}5$ we get a standard Wiener process.

Smoothness of the trajectories of the process $B_H(t)$ is defined by the parameter H: almost all the trajectories satisfy the Holder condition:

$$|X(t) - X(s)| \le c|t-s|^{\alpha},\ \alpha < H,$$

This generalizes known Levy's result for the Wiener process.

The increments of fbm $B_H(t_2) - B_H(t_1)$, $B_H(t_4) - B_H(t_3)$, $t_1 < t_2 < t_3 < t_4$ are form a Gaussian random vector with a correlation between the coordinates:

$$\frac{1}{2}((t_4 - t_1)^{2H} + (t_3 - t_2)^{2H} - (t_4 - t_2)^{2H} - (t_3 - t_1)^{2H})$$

For discrete time:

$$\xi_k = B_H\left(\frac{k}{n}\right) - B_H\left(\frac{k-1}{n}\right),$$

We obtain the correlation coefficient:

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$$\rho(\xi_j,\xi_k)=\frac{1}{2}(|k-j+1|^{2H}+|k-j-1|^{2H}-2|k-j|^{2H}), \quad (2)$$

It means that increments are forming stationary (in the narrow sense) sequence [12].

Let's mention some several properties of fBm:

1. Changing time scale is equivalent for changing of "amplitude" of the process:

$Law(B_H(at))= Law(a^H B_H(t))$,

This equality denotes the coincidence of one-dimensional distributions of the processes:

$B_H(ta)$ and $a^H B_H(t)$

This property is called self-similarity process and it is useful for analysis of time series.

2. Let's put in the formula (2): j=k+n. Then the correlation coefficient:

$$\rho(\xi_k,\xi_{k+n})=\frac{1}{2}((n+1)^{2H}+(n-1)^{2H}-2n^{2H}), \quad (3)$$

$$\rho_n=\rho(\xi_k,\xi_{k+n})\sim H(2H-1)n^{2H-2}, \quad (4)$$

So the memory decreasing for increments has a power character; the increments are independence with $H=\frac{1}{2}$. With $H<\frac{1}{2}$ the increments are form the sequence with short, $H>\frac{1}{2}$ with a long memory. The sign of correlation coefficient ρ_n, which is defined by formula (4), depends form value H: $\rho_n<0,\ H<\frac{1}{2}$, $\rho_n>0,\ H>\frac{1}{2}$. For $H<\frac{1}{2}$ sequence ξ_n of increments fbm is calling pink noise and negativity of variations means the fast variability values. The process of fbm $H<\frac{1}{2}$ is known as anti-persistent. For $H>\frac{1}{2}$ sequence y_n of increments fbm is calling black noise and process of fbm is known as persistent. The properties of persistence have data which are describing some of the physical processes, such as solar activity [13,14].

In this paper, is selected fractional Brownian motion as a basic process.

The Statistics of Fractional Brownian motion

The estimation of Hurst exponent

Let's observe the data:

$$x_k=\sigma B_H\left(\frac{k}{n}\right),\ y_k=x_k-x_{k-1}$$

Let's random vector

$\mathbf{y}=(y_1,y_2,..,y_n)\sim\aleph(0,V)$, where the correlation matrix $V=\frac{\sigma^2}{n^{2H}}S$ and elements S_{jk} of matrix $S\equiv S_H$ is defined by equality (3).

The limit theorems for sequence $y_1,..,y_n$ were first proved by Peltier for statistics [15].

$$R_{jn}=\frac{1}{n}\sum_{k=1}^{n}|y_k|^j,\quad j\in IR$$

$$\mathrm{E}_n(j)=\mathrm{E}R_{jn}=\frac{\sigma^j}{n^{jH}}\frac{2^{\frac{j}{2}}\Gamma\left(\frac{j+1}{2}\right)}{\sqrt{\pi}},$$

With probability 1

$$\frac{R_{jn}}{\mathrm{E}_n(j)}\to 1,\quad n\to\infty \quad (5)$$

From (5) is follows consistency estimates of parameters H,σ:

$$\hat{H}_n=\frac{\ln\left(\sqrt{\frac{2}{\pi}}\frac{\sigma}{R_{1n}}\right)}{\ln n}$$

With known σ:

$$\hat{\sigma}_{1n}=n^H\sqrt{\frac{\pi}{2}}rR_{1n}=1{,}25n^H R_{1n} \quad (6)$$

With known H.

Let's propose new method of estimation Hurst exponent [16-18].

Let's introduce the notation:

$$Q(H)=\frac{0{,}8}{R_{1n}}\sqrt{\frac{(S^{-1}y,y)}{n}}$$

Where matrix $S\equiv S_H$ is defined.

Statement

Statistic

$$\hat{H}=\arg\min|Q(H)-1|$$

Is a consistent estimator of the parameter H.

Proof

ε Is the canonical Gaussian vector with the following characteristics:

$$\mathrm{E}\varepsilon=0,\ \mathrm{E}(\varepsilon,u)(\varepsilon,v)=(u,v)\ \dim\varepsilon=n.$$

Then

$\mathrm{y}=V^{\frac{1}{2}}\varepsilon$, therefore

$$n=\mathrm{E}(\varepsilon,\varepsilon)=\mathrm{E}(V^{-1}y,y)=\frac{n^{2H}}{\sigma^2}\mathrm{E}(s^{-1}y,y)$$

And consequently the statistic

$$\hat{\sigma}_{2n}^2=(n)^{2H-1}(S^{-1}\mathbf{y},\mathbf{y})$$

And here statistics $(n)^{2H-1}(S^{-1}y,y)$ is an unbiased estimate of the parameter σ^2. Let's introduce:

$$\hat{\sigma}_{2n}=\sqrt{n^{2H-1}(S^{-1}\mathbf{y},\mathbf{y})} \quad (7)$$

With calculating the dispersion

$$D\hat{\sigma}_{2n}^2=n^{4H-2}\mathrm{E}(S^{-1}\mathbf{y},\mathbf{y})^2-\sigma^4$$

Use the formula for integration by parts for the Gaussian measure [19] and get

$$D\hat{\sigma}_{2n}^2=\frac{2\sigma^4}{n}\to 0,\quad n\to\infty$$

$\hat{\sigma}_{2n}$ Is a consistent parameter estimation of σ.

The equalities (6,7) are form the system, from which follows the relation:

$$\frac{\hat{\sigma}_{2n}}{\hat{\sigma}_{1n}}=\frac{0{,}8}{R_{1n}}\sqrt{\frac{(S^{-1}\mathbf{y},\mathbf{y})}{n}}\approx 1,\quad \left|\frac{\hat{\sigma}_{2n}}{\hat{\sigma}_{1n}}-1\right|\to\min$$

This proves the statement. The efficiency of proposed estimation method has been tested by numerical experiment [16].

The limit theorems for some statistics

The limit theorems for statistics from increments of fractional Brownian motion have been proved in works of Nourdin I and others [20-23].

$$\xi_k = n^H \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \right) \sim \aleph(0;1)$$

Let's denote

$$\alpha_k = n^H B\left(\frac{k}{n}\right) = \sum_{j=1}^{k-1} \xi_j$$

There is a Mean-square convergence:

$$\frac{1}{n}\sum_{k=1}^{n} \alpha_k \xi_k^3 \to -\frac{3}{2} \quad H \in \left(0;\frac{1}{2}\right), \qquad (8)$$

$$\frac{1}{n^{1+H}} \sum \alpha_k^2 \xi_k^3 \to 3\eta \quad H \in \left(0;\frac{1}{2}\right) \qquad (9)$$

Where

$$\eta \sim \aleph\left(0;\frac{1}{2H+2}\right);$$

$$\frac{1}{n^{2H}}\sum_{k=1}^{n} \alpha_k \xi_k^3 \to \frac{3}{2} B^2(1) \quad H \in \left(\frac{1}{2};1\right), \qquad (10)$$

These results allow us to estimate the adequacy of model with the basic process-fractional Brownian motion.

Construction and Checking the Adequacy of Model

It requires initial analysis of increments $\{y_1,..,y_n\}$ to determine the conversion Φ.

In particular, is it necessary to estimate the one-dimensional distribution of sample and correlation. These actions are possible only with a large sample size (n>5000). We propose new empirical method of transformation increments $\{y_1,..,y_n\}$ in $\{z_1,..,z_n\}$ for small sample.

The first stage of approximation is an empirical method for testing the hypothesis about normality increments of model. The criterion may be Gaussian value "kurtosis" (excess kurtosis):

$$d_n = \frac{\left(\frac{1}{n-1}\sum_{k=2}^{n} | y_k |\right)^2}{\frac{1}{n-1}\sum_{k=2}^{n} y_k^2},$$

Which is equal $\frac{2}{\pi}$ for Gaussian model. If d_n is significantly different from $\frac{2}{\pi}$, let's replace the time series $y_1 \text{-} y_{n-1}$ by the new sequence $z_1 \text{-} z_{n-1}$.

The general idea of approximation is an one-dimensional functional transformation g of each increment y_k, where g is an increasing odd function,

$$z_k = g(y_k)$$

Let's assume

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n-1}\sum_{k=1}^{n-1} |g(y_k)|\right)^2}{\frac{1}{n-1}\sum_{k=1}^{n-1} (g(y_k))^2} = \frac{2}{\pi},$$

Where $z_k = g(y_k)$ is assumed as a Gaussian random value. Let's demonstrate proposed algorithm with g as a power function. Assume

$$z_k = \operatorname{sgn} y_k |y_k|^{\frac{1}{\lambda}} = \Phi^{-1}(y_k) \quad \text{.......................... (11)}$$

$$y_k = \operatorname{sgn} y_k |z_k|^{\lambda} = \Phi(z_k) \;\; \lambda > 0, \quad \text{.................... (11A)}$$

Then

$$d_n = \frac{\left(\frac{1}{n-1}\sum_{k=1}^{n-1} |z_k|^{\lambda}\right)^2}{\frac{1}{n-1}\sum_{k=1}^{n-1} |z_k|^{2\lambda}}$$

$d = \lim_{n\to\infty} d_n$ Is equal to ratio of the corresponding mathematical expectations. For $\xi \sim \aleph(0;\sigma^2)$

$E|\xi|^{\alpha} = \frac{2^{\frac{\alpha}{2}}}{\sqrt{\pi}} \sigma^{\alpha} \Gamma\left(\frac{\alpha+1}{2}\right)$, then $d = \frac{1}{\sqrt{\pi}} \frac{\Gamma^2\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)}$

Where the parameter **λ** is defined from the equation. Thus, the proposed approximation leads to the following model of original time series:

$$x_k = \sum_{j=1}^{k} \operatorname{sgn} y_j \cdot | z_j |^{\lambda}$$

If we'll assume that values of sequence $\{z_{1,..,} z_n\}$ are increments of fBm, let's calculate Hurst exponent by the following algorithm, which shows proposed method:

1) Construct the statistic:

$$R_1 = \frac{1}{n}\sum |z_k| = |\bar{z}|$$

2) Calculate the matrix S_H^{-1}, where S_H is a correlation matrix of increments:

$$s_{jk} = E\left(B_H\left(\frac{k}{n}\right) - B_H\left(\frac{k-1}{n}\right)\right)\left(B_H\left(\frac{j}{n}\right) - B_H\left(\frac{j-1}{n}\right)\right) = \rho(\xi_j, \xi_k)$$

$$Q = \frac{0{,}8}{R_1}\sqrt{\frac{(S_H^{-1}\mathbf{z},\mathbf{z})}{n-1}},$$

The statistics Q is calculating for difference values of Hurst exponent with step (0, 0.5-1) and:

$$|Q(H) - 1| \to \min, \;\; \hat{H} = arg\,min|Q(H) - 1| \qquad (12)$$

3) The testing of hypothesis T= (statistics $z_1,..,z_n$ which obtained by transformation (11) of real data are simulated by increments from fractional Brownian motion). The algorithm with known H is the following. Denote

$$c = \frac{1}{n}\sum_{k=1}^{n} z_k^2$$

And assume that hypothesis T is done

$$z_k = \sqrt{c}\,\xi_k = \sqrt{c}\, n^H \left(B\left(\frac{k+1}{n}\right) - B\left(\frac{k}{n}\right) \right)$$

Assume $v_k = \sum_{j=1}^{k-1} z_j$ and construct the statistics

$$A_n = \frac{1}{n}\sum v_k z_k^3, \quad H \in \left(0;\frac{1}{2}\right);$$

$$B_n = \frac{1}{n^{1+H}} \sum v_k^2 z_k^3, \quad H \in \left(0; \frac{1}{2}\right);$$
$$D_n = \frac{1}{n^{2H}} \sum v_k z_k^3, \quad H \in \left(\frac{1}{2}; 1\right) \quad (13)$$

If hypothesis T is true, there is convergence:

$$A_n \to -\frac{3}{2}c^2; \; B_n \to 3c^{\frac{5}{2}}\eta; \; D_n \to \frac{3}{2}c^2 B^2(1)$$

The decision about hypothesis T is accepted by comparing the real values of the statistics with their theoretical limit values. Let's determine deviation from the limit values for statistic A_n.

$$\delta = \left|\frac{A_n - A}{A}\right|$$

The limit distribution functions for statistics B_n, D_n

$$F_1(x) = P\{3\,c^{2,5}\eta < x\} = \Phi(\frac{x}{3\sigma}c^{-2,5}), \; F_2(x) = 2\Phi(\frac{1}{c}\sqrt{\frac{2}{3}}x) - 1, x > 0,$$

Where Φ is Laplace function, $\sigma = (2H+2)^{-0,5}$

The hypothesis T is accepted, if:

$$\delta < \beta_0, |B_n| < \beta_1, H < 0{,}5: \qquad 0 < D_n < \beta_2, H > 0{,}5 \quad (14)$$

Where β_1, β_2 are quantile distributions from F_1, F_2 which corresponding to the selected significance level $\alpha = 0{,}1$.

$$\beta_1 = \frac{4{,}95c^{2,5}}{\sqrt{2H+2}}, \beta_2 = 4{,}08c^2, c = \overline{z}^2$$

The Real Examples

Let's consider examples of real nature:

The first example: the monthly data of market rate of the Bundesbank (Germany) (http://www.bundesbank.de) for 2003-2012 (120 data) (Figure 1).

The second example: 1020 data of exchange rate EUR / USD for 2011-2014 (http://www.banque-france.fr) (Figure 2).

The third example: The oscillation of waves in the North Atlantic, 10.1980-10.2014 409 data (Figures 3 and 4).

The Comparative Analysis of Used Models

Let's compare the time series model (approximation of fractional Brownian motion) with known models and estimate the quality of modeling. Note that the choice of the quality criterion is dependent from the type of model.

The values of exchange rate, Banque de France. Let's compare the effectiveness of approximation method with other models for real 1020 data [21].

For modeling of selected values are used these models:

- Autoregression,

Autoregression with moving average (ARMA) (p,q),

Autoregressive with integrated moving average (p,d,q),

Autoregressive moving average (ARMA) (p,q).

These methods have been selected, because after using the special tests for statistical data, we've revealed high autocorrelation value and existence of a trend.

Based on analysis of values of the constructed partial autocorrelation and autocorrelation function of data series, the order AR (1) model may be in the range from 1 to 5. The model AR (1) is given:

$$y(k) = a_0 + a_1 y(k-1) + \varepsilon(k) = 0{,}101 + 0{,}908\,y(k-1) \quad (15)$$

Where $y(k)$ is a basic variable; $\varepsilon(k)$ is a random process. Characteristics of the adequacy and quality for short-term forecasts for the training sample had the following values:

$R^2 = 0{,}816$, $\sum e_k^2 = 52{,}591$, $DW = 1{,}957$, Mard=10,71, $U = 0{,}069$

Some deterioration of forecasting is obtained by expansion of the order of autoregression for two:

$$y(k) = a_0 + a_1 y(k-1) + a_2 y(k-2) + \varepsilon(k) =$$
$$= 1{,}176 + 1{,}0047\,y(k-1) - 0{,}111\,y(k-2) \quad (16)$$

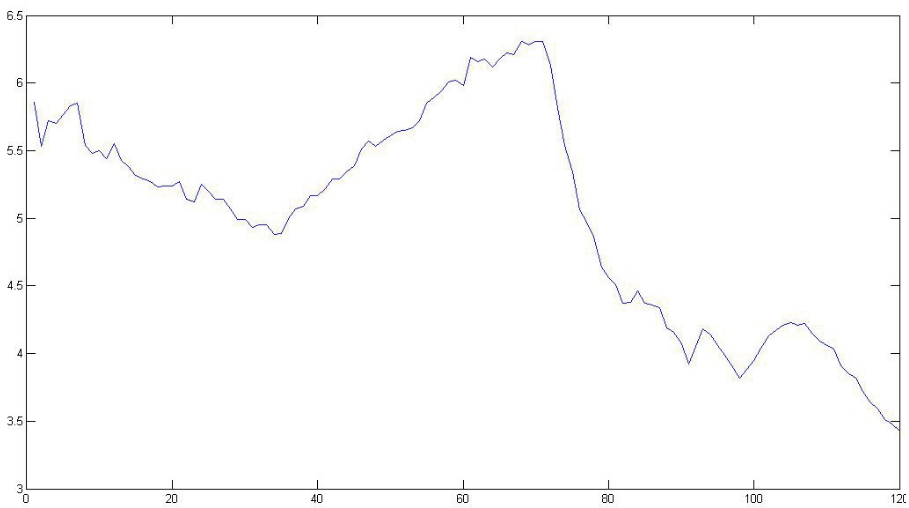

Figure 1: The monthly data of market rate of the Bundesbank (Germany).

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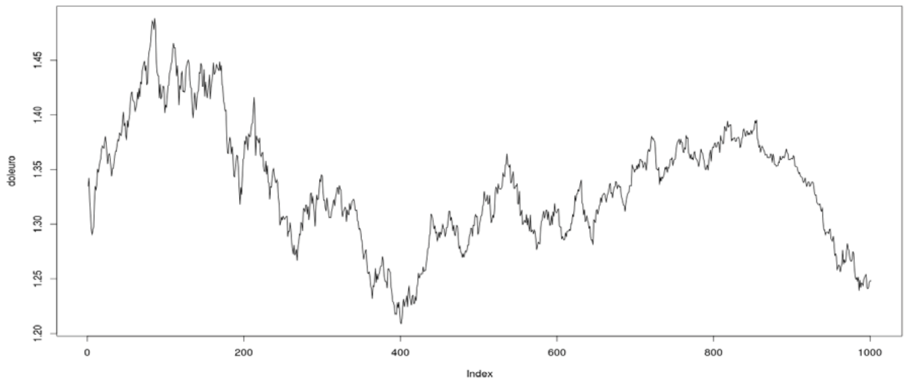

Figure 2: The data of exchange rate EUR/USD, Banque de France.

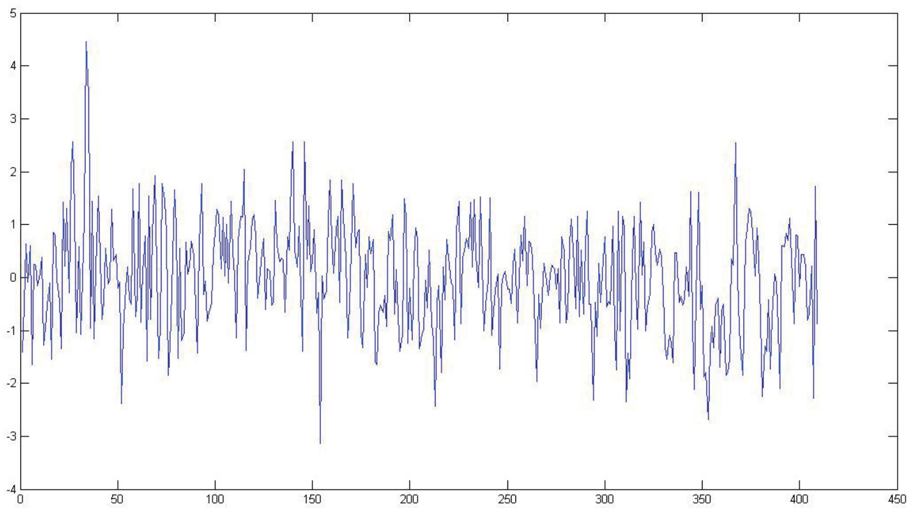

Figure 3: The oscillation of waves in the North Atlantic.

Model	R^2	$\sum e_k^2$	DW	MARD	Coefficient Teil
AR (1)	0,816	52,591	1,785	10,713	0,069
AR (5)	0,820	50,771	1,957	11,197	0,071
AR (5) «plus» cubic trend	0,844	43,842	1,720	5,783	0,036
Approximation method	—	—	1,987	5,691	—

Table 1: Comparative characteristic of models. Exchange rate data, Banque de France.

R^2=0,820, $\sum e_k^2 = 50.771$, DE=1,57 , MARD=11,197, $U = 0{,}071$.

$$y(k) = a_0 + a_1 y(k-1) + a_5 y(k-5) + a_6 k + a_7 k^2 + a_8 k^3 + \varepsilon(k) =$$
$$= 3{,}957 + 0{,}743\, y(k-1) + 0{,}005\, y(k-5) -$$
$$-0{,}035k + 0{,}00036k^2 - 1{,}24E - 0{,}6k^3 \quad (17)$$

$R^2 = 0{,}844$, $\sum e_k^2 = 43{,}841$, $DW = 1{,}720$, $\tilde{N}\hat{E}\ddot{I} = 0{,}814$, $\tilde{N}\grave{A}\hat{I}\hat{I} = 5{,}783$, $U = 0{,}036$

Thus, mean absolute relative difference has been reduced from 11.20% to 5.78% after introducing the trend in model. Let's construct a model with autoregressive moving average:

$$y(k) = a_0 + a_1\, y(k-1) + a_2 y(k-5)\; b_1\, \varepsilon(k-1) + b_2\, \varepsilon(k-2) + \varepsilon(k) =$$
$$= 10.82 + 0{,}757\, y(k-1) + 0{,}134\, y(k-5) +$$
$$+ 0{,}209\, \varepsilon(k-1) + 0{,}271\, \varepsilon(k-2) \quad (18)$$

However, its characteristics aren't better than in the previous case (the model with the trend), except for Durbin-Watson statistic (Table 1):

$R^2 = 0{,}832, \sum e_k^2$ =4,6931, $DW = 1{,}943$, MARD=10,611, $U = 0{,}068$

Thus, the best is a structural model of the process from all constructed mathematical models, which takes into account explicitly the trend of process and vibrations (MARD=5.78%) [17-19]. This is quite a logical result, because the structural models are describing these processes with a higher degree of adequacy than others. As expected, the introduction of a moving average model didn't improve the quality as compared with a simple model AR (1). The value of the Durbin-Watson statistic for the approximation model is more closer to 2 than

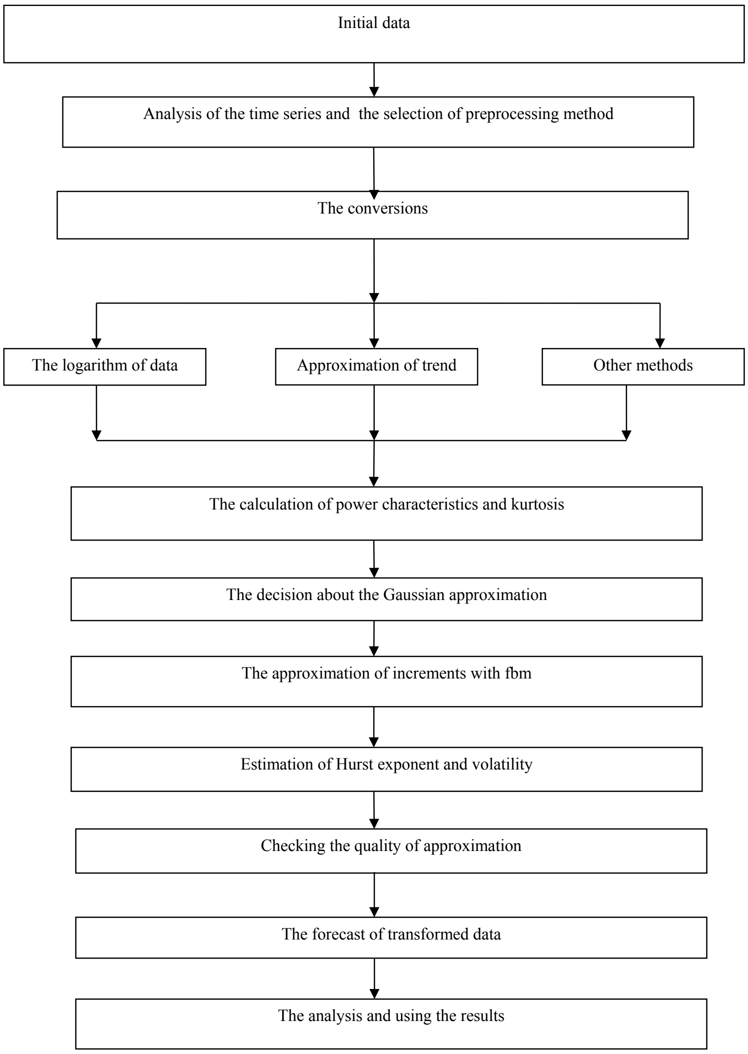

Figure 4: The block diagram.

Real data	$\hat{H}$	A_n	B_n	A	β_1	β_2
The monthly data of Bundesbank	0,4	-19,1	-0,8	-16,3	58,3	—
Exchange rate €/$,2011-2014, Banque de France	0,4	-1,72	-1,61	-1,51	5,28	—
The oscillation of waves in the North Atlantic.	0,1	-3,44	-6,38	-3,575	9,883	—

Table 2: The values of control statistics and parameters of limit distributions for Examples 1-3.

the other models, the value of MARD is practically coincides with its value for the AR (5) "plus" cubic trend.

We consider the primary data processing as a calculation of linear approximation of the trend of initial data and obtaining new sequence in every example $\{x_k\}$, $\bar{x} = 0$.

The following steps are calculation the increments $y_k = x_{k+1} - x_k$, construction the new sequence $\{z_1,..,z_n\}$ by formula (11), the estimation of Hurst exponent by formula (12), checking the quality of approximation by formula (10). The results of calculations are shown in Table 2.

Conclusion

For all examples the approximation has antipersistent character (H<0.5) and it's adequate, if the conditions are satisfied [14].

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