GLOBAL JOURNAL OF INTERDISCIPLINARY SOCIAL SCIENCES
(Published By: Global Institute for Research & Education)
www.gifre.org

Misinterpreting the Meaning of the Equal Sign: A Study of Pupils in the Final Grades of Primary School in Cyprus

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Abstract

The objective and originality of this study was the application of a multi-faceted approach to the investigation of the understanding of the equal sign in primary school children. In the first phase of the study a large proportion of a sample of 126 final grade pupils displayed difficulties in interpreting the concept. In the second phase, all of the primary school mathematics textbooks were reviewed in terms of how they present equalities; a heavy emphasis on the typical operations-equals-answer context was found. In the final phase a sample of 226 children from the final two grades of primary schools was selected. Brief intervention with instruction on the relational meaning of the equal sign, through a specifically designed worksheet containing equations in the non-typical context and some word problems, was found to have a significant positive impact on children's understanding after a reasonable lapse of time.

Keywords: equal sign, interpretation, primary schools.

1. Introduction

Equality plays a significant part in our everyday life. Each individual's concept of equality begins to form in childhood and continues to develop until well into adulthood. Kieran (1981) argues that children's understanding of equality and the equal symbol is distinct from that of an adult. Adults look upon equality sentences as equivalence relations while children at the elementary level interpret the equality symbol as an indication to find the total.

1.1 Difficulties in interpreting the equal sign

It is well documented that elementary school children (aged 7-11) in the United States have substantial difficulty learning to interpret the equal sign as a relational symbol of mathematical equivalence (Behr, Erlwanger & Nichols, 1976; Kieran, 1981; Li, Ding, Capraro & Capraro, 2008). Many view the equal sign operationally, as a command to carry out arithmetic operations, rather than relationally as an indicator of equivalence (Rittle-Johnson, Mathews & Taylor, 2011; Stephens, Knuth, Blanton, Isler, Gardiner & Marum, 2013). Falkner, Levi and Carpenter (1999) argue that most pupils in elementary schools when solving equations such as $8 + 4 = \dots. + 5$, either add the numbers before the equal sign or add all the given numbers giving answers such as 12 or 17.

Kieran (1981) suggested that difficulties in understanding the equal sign extended beyond primary schooling and may be responsible for many errors encountered in secondary and post-secondary mathematics. McNeil, Grandau, Knuth, Alibali, Stephens, Hattikudur & Krill, (2006) reported that some middle-school students (ages 11-14) continue to interpret the equal sign as an operational symbol, despite being developmentally ready to interpret it as a relational symbol.

1.2 The developmental cognitive limitations suggestion

McNeil et al. (2006), in citing Piaget and various colleagues, mention that children's difficulties may be due to developmental cognitive limitations. They then continue by adding that if this is the case, then children may not be developmentally ready to learn the relational concept before a certain age. Why then should teachers spend valuable class time trying to teach children something that they are not ready to learn? Instead, teachers should just wait until children are old enough to learn the relational concept and teach it to them at that point.

However, it does not appear as though children's difficulties can be attributed to domain general conceptual limitations in childhood. Indeed Rittle-Johnson and Alibali (1999) have shown that children in this age range succeed when asked to identify which pair of numbers (e.g. 3 + 4, 7 + 1, 5 + 6) is equal to a given pair (e.g. 5 + 3) suggesting that they can identify an equivalence relation between two added pairs. Moreover, difficulty in understanding equivalence is not universal. The results of Chinese pupils of the same ages (98% exhibited a relational understanding of the equal sign, as opposed to 28% of the U.S. pupils) suggest that age-related conceptual limitations are not the primary source of children's difficulties (Li et al., 2008).

1.3 The learning context

> … if difficulties with the equal sign are due to knowledge built from early experience with arithmetic, then students' ability to acquire the relational concept of the equal sign may depend on the learning context. If this is the case, then teachers can work to improve aspects of the learning context to promote the relational concept. (McNeil et al. 2006, p. 369)

For example, instead of the standard operations-equals-answer equation context, teachers could present equations in non-standard contexts that highlight the equal sign as expressing an equivalence relationship (e.g. $3 + 4 = 5 + 2$).

Unfortunately, children develop ideas about addition and subtraction before entering school (Seo & Ginsburg, 2003) and they experience early elementary school instruction that exclusively involves equations in standard form (operation-equals-answer form, such as 3 + 4 = ….). “Such heavy focus on standard equations in textbooks and school work leads students to an operational understanding of the equal sign” (Powell and Fuchs 2010, p. 383), and to interpreting the equal sign as a signal to perform the operations preceding it.

There seems to be a general agreement that perhaps the most significant cause of children’s difficulties with mathematical equivalence is specific knowledge that children construct from their early experience with arithmetic in schools (Carpenter, Franke & Levi, 2003; McNeil & Alibali, 2005a; Seo & Ginsburg, 2003). Notably, McNeil and Alibali (2005a) suggested that children’s reliance on their knowledge of these operational patterns (operations-equals-answer forms) may hinder their ability to even benefit from brief one-to-one intervention in math equivalence problems. Carpenter, Franke and Levi (2003) and Falkner, Levi and Carpenter (1999) also advocate that the difficulty that U.S. children have in understanding mathematical equivalence stems from their prior experiences with the equal sign. They receive little explicit instruction on the meaning of the equal sign. Consequently, they infer an incorrect meaning of the equal sign from repeated experience with limited equation structures.

Many researchers have concluded that one of the requirements for being ready to handle algebra is “a much richer understanding of the equal sign than that which is provided by traditional arithmetic” (Freiman & Lee 2004, 416). Knuth et al. (2006) argue that at the start of secondary schooling “a relational view of the equals sign is necessary not only to meaningfully generate and interpret equations but also to meaningfully operate on equations” (p. 309).

1.4 Content of textbooks

Seo and Ginsburg (2003) reviewed two second-grade textbooks and identified very few instances where the equal sign was not presented in an operation-equals-answer form. Falkner, Levi and Carpenter (1999) argued that there is no variety in the way the equal sign is presented in elementary schools (implying in the textbooks) and add that

> Usually the equal sign comes at the end of an equation and only one number comes after it. With number sentences such as 4 + 6 = 10 or 67 – 13 = 54, the children are correct to think of the equal sign as a signal to compute. (p. 232)

McNeil et al. (2006) reviewed four popular middle-school mathematics textbooks and reported that they frequently present the equal sign in an operations-equals-answer context and rarely present the equal sign in an operations-on-both-sides context. They concluded that middle-school mathematics textbooks may not be optimally designed to help students acquire a relational understanding of the equal sign.

Baroody and Ginsburg (1983) carried out a survey with children of first, second and third grades. The students used the Wynroth mathematics curriculum in which the equal sign was defined as a literal translation “the same as” which gives a relational meaning of equal sign without using the phrase “the answer is”. Also the Wynroth curriculum used the operations-on-both-sides form in a variety of ways such as: $9 + 3 = 5 + 7$; $4 + 2 < 3 + 5$; $8 + 7 \neq 5 + 6$. First graders used the Wynroth curriculum for seven months as did the second and third graders. Second and third graders had previously been taught from the traditional textbook and this was found to cause a significant conceptual difference between first and second graders, with the first graders exhibiting a better understanding than the second graders who had been influenced by the traditional approach.

McNeil (2008) reported that the presence of typical arithmetic problems hinders learning of mathematical equivalence, because they activate overly narrow representations that do not facilitate broad generalisation. Specifically, typical arithmetic problems are thought to activate children’s representations of first, the “perform all the given operations” problem-solving strategy; second, the “operations-equals-answer” problem format; and third, the “calculate the total” interpretation (McNeil & Alibali 2005b, p. 884).

1.5 Value of appropriate instruction

In the absence of appropriate instruction, approximately 70% of children solve mathematical equivalence problems incorrectly (Li et al., 2008) and, even after receiving instruction, many children continue to have difficulty (Rittle-Johnson and Alibali, 1999). Wolters (1991) concluded that students’ mastery of the concept of equivalence was significantly related to the type of instruction they received. Powell and Fuchs (2010) also reported that explicit equal-sign instruction positively impacts understanding of the relational meaning of the equal sign for students with mathematical difficulties, and this better understanding transfers to word-problem solving. In China, for example, textbooks and guidebooks use multiple representations for equivalence and as a result Chinese pupils can interpret the equal sign as a relational symbol of equivalence (Li et al., 2008).

It seems reasonable to suggest that the contexts in which teachers, curricula and textbooks present the equal sign play a major role in the development of students’ understanding of the equal sign.

1.6 Calculators

Calculators also play a role in the aggravation of the misunderstanding of the equal sign, according to Van de Walle (2004). Children write the operation on the calculator and then press the button ‘=’ in order to see the answer. In their mind the equal symbol means ‘the answer is’ because usually the equal symbol comes on the right of the equation and generally only one number follows. Thus, “calculators arguably contribute significantly to children’s adoption of the operator meaning” (Jones & Pratt, 2005, p. 185).

1.7 The study

Over the last three decades research has suggested that children’s difficulties with mathematical equivalence are caused, at least in part, by specific knowledge that children construct from their narrow experience with typical arithmetic problems in school (Carpenter, Franke & Levi, 2003; McNeil, 2008). Therefore, given that pupils’ interpretations of the equal sign are likely to be formed by context, it is important to examine the contexts in which pupils typically encounter the equal sign.

To this end, and as a result of the researcher's experience as a maths teacher in high schools in Cyprus where he sometimes encounters misconceptions of the equal sign even at this level, the goal of the present study was to provide an original multi-faceted, thus more complete, approach to investigating the misinterpretation of the equal sign among primary senior pupils. It first explores the extent of any possible problem in misinterpreting the equal sign among sixth graders (age 11), in a sample of pupils from Limassol, a town in Cyprus. Second, all the elementary school maths textbooks (grades one to six) were reviewed in order to investigate the context in which the equal sign is presented. Finally, an investigation was carried out to determine whether receiving a brief intervention, in terms of specifically designed instruction to teach the relational meaning of the equal sign in equations, would have a positive impact on the children's understanding of the concept after a reasonable lapse of time.

2. Methodology

Given the multi-faceted purpose of the study, it was divided into three phases. In the first phase, which took place in November 2010, a test comprising of simple equalities and short problems was developed using similar items to those used by Li et al. (2008). The researcher believes that these items were appropriate as they covered the concept well. Since the emphasis of the test was on the relational meaning of the equal sign, and not on the ability of pupils to carry out difficult operations, simple numbers were used and simple word problems of the kind that pupils had previously encountered many times in their school years. The researcher presented the set of items to a group of four experts (experienced junior high school maths teachers) for an evaluation of appropriateness. They all agreed that those items were indeed suitable for investigating the relational meaning of the equal symbol.

The test was administered to children in seven sixth grade classes in four different, randomly selected, primary schools in the town of Limassol. The purpose of the study was thoroughly explained to the headteachers, teachers and pupils and their consent to carry out the study was received.

The test was administered by the researcher to all classes, in the presence of their teachers, and it took approximately 30 minutes to complete. The sample consisted of 126 children (66 boys and 60 girls). Following the analyses, the results were presented to the teaching staff of the four primary schools during one of their regular staff meetings in June 2011. In turn, the teachers conveyed the results to their pupils.

Having determined the existence of the problem, the researcher sought to investigate the reasons behind it. Given that children's interpretations of the equal sign are likely to be shaped by context, it is important for researchers to investigate the contexts in which pupils typically come across the equal sign. This led to the second phase of the study, which took place in the summer of 2011, in which the presentation of the equal sign in the textbooks of all grades of the elementary school was explored. The researcher examined the proportion of equal sign instances presented in the standard operations-equals-answer context as this context is thought to promote an operational interpretation of the equal sign. The researcher was also interested in the extent to which the textbooks present the equal sign in the operations-on-both sides and operations-on-the-right-hand-side contexts, because these contexts have been shown to elicit the relational interpretation of the equal sign (McNeil & Alibali, 2005a).

It should be noted that in the Cypriot educational system the Ministry of Education and Culture prepares all mathematics' textbooks and these are used mandatorilly throughout the country by all primary schools. Each year group's textbook comprises of four volumes with the exception of the second grade's which has five volumes. All the textbooks were reviewed by the researcher. Reliability of the reviewing process was established by having a second, independent, reviewer evaluate one randomly selected volume from each of the six sets of books. The degree of agreement between the independent reviewer and the researcher was close to 100% (98.9%).

Having determined the existence of the problem and examined the context of instruction through the textbooks the researcher sought to determine the possible impact of a brief intervention. In the third phase, in early September of 2011, the researcher presented the findings of the textbook reviews to three of the four primary schools involved in the first phase, those where teachers had agreed to participate again. Since this phase took place one year later, it should be noted that while some of the teacher participants were the same as in the first phase, the pupils were new cohorts. The researcher, in collaboration with another high school mathematics teacher, prepared a specially designed worksheet with guidelines and examples on the relational interpretation of the equal sign (see Appendix 1 for the translated version of the worksheet). Emphasis was placed on the operations-on-both-sides context. The aim was not to teach the relational meaning of the equal sign in detail but merely to introduce the children to it, through the worksheet, and to investigate whether that would have a positive impact on their understanding after a reasonable lapse of time. This time one (out of two available) sixth and one (out of two available) fifth grade classes in each of the three schools were randomly selected to receive instruction, by their own teachers, on the equal sign following the directions of the worksheet.

2.1 Guidelines to teachers regarding the administration of the worksheet

The researcher arranged three group meetings, one in each school, with the four teachers involved. These meetings took place twice, once before and once after the administration of the worksheet. During the first meetings the teachers of the selected classes were advised to spend two to three 40-minute periods on the worksheet, and to then wait for a period of four weeks to elapse before administering the test without any warning or any further instruction on the equal sign. They were given the following guidelines to follow during the administration of the worksheet.

First, ensure that all pupils are familiar with the four operations with simple numbers. This can be checked with exercise 1 on the worksheet. Then explain the relational meaning of the equal sign and ask pupils to read and copy, into their exercise books, what is written on the computer screen they have in front of them on the first page of the worksheet.

Exercise 2 is perhaps the most important as it contains equations in all contexts, with special emphasis on the non-standard ones. The researcher explained to the teachers the necessity to remind the pupils that the result of all the operations on the one side must be the same as the result of all operations on the other and to illustrate this by calculating separately, where applicable, the result on both sides of the equations before commencing this exercise, as well as any other exercise.

For exercises 3 and 4 which present word problems associated with equations it is important for pupils to understand how a word problem can be solved through the use of an equation. The numbers and operations are very simple so emphasis should be placed on writing the correct equation. Exercise 5 consists of a problem and a suggested solution which is mathematically incorrect, even though the final result is correct. The suggested solution is 15 + 12 = 27 x 2 = 54. Here emphasis should be placed on clarifying to the pupils that once the equal sign has been used what follows must be the same as the result of all operations preceding the equal sign.

And finally exercise 6 presents a similar kind of problem to one which is frequently used in the first grade of junior high school to introduce the idea of solving equations in algebra. In this exercise the relational meaning of the equal sign is introduced to the pupils with the help of weighing scales.

In order to avoid possible uneasiness or interference in the normal teaching practices within the primary school classrooms, the researcher did not observe this process.

During the second meetings with the three groups, the teachers confirmed that they had followed all guidelines implicitly and the vast majority of the pupils responded positively to the instruction they were given on the equal sign. The researcher also enquired as to whether it was common practice in the Cypriot primary schools to follow the textbooks to the letter. The teachers responded affirmatively adding also that since the math textbooks were prepared centrally, by the Ministry of Education, to cover the exact curriculum, they felt obliged to follow the textbooks to the letter. They added that they occasionally do provide extra work for their pupils. In the case of the equal sign since there was no specific reference to this concept in the textbooks, it was the first time any of them had discussed it with their pupils.

2.2 The test in phase 3

Two more equations were added to the original test from phase 1 to facilitate scoring, giving a possible total score of 20. The test was administered to six sixth grade classes and six fifth grade classes in the three schools. The sample was made up of two groups: 117 pupils of three fifth and three sixth grade classes who had received specific instruction regarding the relational meaning of the equal sign (the experimental group), and an equivalent group of 109 pupils of three sixth and three fifth grade classes who had not (the control group). The sample consisted of 226 children (120 boys and 106 girls). Each question for which the pupils had to find the answer carried one mark and the four word problems carried 2 marks each. As in the first phase, the purpose of the study was thoroughly explained to all involved and their consent was received.

3. Results

3.1 Phase 1 – First administration of the test

Table 1 shows the first test items and the responses of the pupils to these, together with the corresponding percentages. The table is divided into four parts. The first part contains one multiple-choice and three matching questions, the correct answer and the percentage of pupils who responded correctly. The second part contains six equations with a missing number, which the pupils had to find. The second column in this part contains the correct answer, the third the percentage of correct responses. The fourth and sixth columns list the most common wrong answer given (Answer A) and the second most common (Answer B). The fifth and seventh columns present the corresponding percentages for these answers. The third part of the table contains the responses of the pupils to the two questions where they had to transform the word problem into an equation. Finally, the last part contains the responses to the questions where pupils were asked to say whether (mathematically incorrect) equations, leading to the right answer, were correct.

Table 1. Responses to the first test

Part 1	Correct	%				
27 + 46 = … (multiple choice)	73	97.6%				
14 – 6 = ……. (matching)	8	92.0%				
9 + 7 = ……. (matching)	16	99.0%				
8 x 7 = ……. (matching)	56	94.2%				
Part 2	**Correct**	**%**	**Answer A**	**%**	**Answer B**	**%**
17 + 20 = …….	37	98.4%				
……. = 38 + 10	48	81.0%	none	7.1%	38	4.8%
13 + 51 = 51 + …….	13	51.6%	115	35.7%	64	2.4%
12 + ... = 28 + 4	20	57.1%	16	16.7%	32	4.8%
160 = ……. – 30	190	69.0%	130	16.7%	none	7.9%
38 – 12 = ……. – 3	29	35.7%	26	27.8%	23	6.3%
Part 3	**Correct equation**		**Wrong equation**		**Solved it**	
Problem 1: Form an equation	50.8%		23.0%		19.0%	
Problem 2: Form an equation	43.7%		31.0%		14.3%	
Part 4	**Yes**		**No**		**No response**	
Is 10 – 4 = 6 – 2 = 4 x 2 = 8 correct?	96.0%		0%		4%	
Is 3 + 8 = 11 x 6 = 66 – 5 = 61 correct?	97%		0%		3%	

In the multiple choice and the matching questions more than 92% of pupils found the correct answers. In the first question (in part 2 of the table), in which the pupils had to complete the missing number in the equation 17 + 20 = ……… 98.4% of them found the correct answer. This is a typical (operations-equals-answer) equation. In the remaining five equations, which were not in the typical form the percentage of correct answers varied from 35.7% to 81%. The highest percentage of correct answers (81%) was obtained in the equation ……. = 38 + 10, whereas the lowest percentage in 38 – 12 = ……. – 3. Common examples of pupils assuming an operational interpretation of the equal sign include:

- giving 115 for 13 + 51 = 51 + ……. (35.7%)
- giving 16 for 12 + ... = 28 + 4 (16.7%)
- giving for 160 = ……. – 30 and (16.7%)
- giving 26 for 38 – 12 = ……. – 3 . (27.8%)

Problems were also evident in the pupils' ability to transform word problems into equations with only about 51% and 44% being successful in the first and second problems respectively.

Finally, the last two questions contained simple problems again, followed by a suggested equation leading to the correct answer. The pupils were asked to say whether, in their opinion, the equation was correct or incorrect (by underlining YES or NO accordingly). In the latter case they were asked to write the correct equation. In both cases the equations included misuse of the equal sign and were mathematically incorrect, even though the answer in the end was correct.

Problem 1: I had €10 and went to the bookshop. I paid €4 for a book and €2 for an exercise book. When I returned home my father asked me how much change I had and he doubled that amount for me. How much money did I eventually have?

Suggested equation: 10 – 4 = 6 – 2 = 4 x 2 = 8

Problem 2: Add 3 to 8, multiply the sum by 6 and then subtract 5 from the result.

Suggested equation: 3 + 8 = 11 x 6 = 66 – 5 = 61

Almost every pupil, 96% in the first and 97% in the second question underlined YES, agreeing that the given equation was right and the remainder, 4% and 3% respectively did not respond. None of the pupils realised that the equation was mathematically incorrect.

3.2 Phase 2 – Reviewing the textbooks

The textbooks used in Cyprus are centrally prepared by the Ministry of Education and Culture and are both mandatory and the same for all primary schools. The researcher reviewed all maths textbooks used in all grades. Table 2 shows the results of this review. The first column shows the grade, the second the number of volumes (and pages) in each textbook and the third the number (and percentage) of standard equalities (operations-equals-answer form). Under non-standard equalities there are three columns. The first shows the number (and percentage) of equalities with operations on both sides; the second with operations on the RHS only and the third other equalities. Finally, the last two columns show the total number of equalities and the mean number per page in each textbook.

Table 2. Results of the maths textbook reviews

Grade	Volumes (pages)	Standard Equalities	Non-standard equalities			Total	Mean per page
			Both Sides	RHS only	Other		
1st	5 (538)	549 (91.3%)	0 (0.0%)	13 (2.2%)	39 (6.5%)	601	1.1
2nd	4 (478)	1017 (97.7%)	7 (0.7%)	0 (0.0%)	17 (1.6%)	1041	2.2
3rd	4 (480)	524 (91.9%)	17 (3.0%)	8 (1.4%)	21 (3.7%)	570	1.2
4th	4 (508)	635 (78.0%)	7 (0.9%)	30 (3.7%)	142 (17.4%)	814	1.6
5th	4 (406)	524 (69.7%)	22 (2.9%)	39 (5.2%)	167 (22.2%)	752	1.9
6th	4 (377)	623 (70.2%)	56 (6.3%)	66 (7.5%)	142 (16.0%)	887	2.4
Total	25 (2787)	3872 (83.0%)	109 (2.3%)	156 (3.3%)	528 (11.3%)	4665	1.7

A few points in Table 2 are noteworthy. During their primary school years in Cyprus, children see 4665 equalities in their maths textbooks (mean of 1.7 equalities per page). Out of these, 3872 (83%) are in the standard operations-equals-answer form. Children only come across the operations-on-both sides form 109 times (2.3%). Out of these 109 cases, only 31 appear in the textbooks for the first four grades (none in the first and only seven times in the second grade's textbook) and this amounts to approximately 1% of all the equalities in these textbooks.

Children also only come across the operations-on-the-RHS form 156 times (3.3%). Out of these 156, only 51 appear in the textbooks for the first four grades (13 in the first, none in the second and eight in the third grade's textbooks) and this amounts to approximately 1.7% of all the equalities in these textbooks.

3.3 Phase 3 – Second administration of the test

Table 3 shows the comparison of the mean scores of pupils in the two groups (experimental and control) on the second test. The result is highly significant ($p < 0.001$) with pupils in the experimental group outperforming (mean score

13.15) those in the control group (mean score 10.88). In the statistical tests performed p-values are reported correct to three decimal places.

Table 3. Comparisons of mean scores of the two groups

Worksheet	N	Mean	S.D.	t-statistic	d.f.	p-value
NO	109	10.88	3.23			
YES	117	13.15	4.10	4.644	224	0.000

The test was then divided into three parts. Part A contained the first four items, three of which were matching and one multiple choice. Part B contained eight items with equations (one in the standard operations-equals-answer form and seven in non-standard forms) where the students had to complete them by entering the missing number. Part C contained four word problems in two of which the pupils had to write the correct equation. In the last two they were given a mathematically incorrect equation, leading to the correct answer, and they had to say whether the equations given were correct or incorrect.

Further comparisons of mean scores of the two groups of pupils were carried out separately for each of the three parts. Table 4 shows the results of these tests. In part A, where no equal signs were involved, the mean scores were identical (3.830, out of a possible maximum of four marks, for the control group and 3.829 for the experimental group). However, significant differences ($p < 0.001$), in favour of the experimental group, were found in comparisons of the mean scores in parts B and C. More specifically, in Part B pupils of the experimental group had a mean score of 6.017 (scoring 75.2% of the total marks), in contrast with 4.817 for the pupils in the control group (scoring 60.2% of the total marks). Similarly, in part C pupils of the experimental group had a mean score of 3.308 (scoring 41.4% of the total marks), whereas pupils in the control group 2.234 (scoring 27.9% of the total marks).

Table 4. Comparisons of mean scores of the two groups for the three parts

PART A	Worksheet	N	Mean	S. D.	d.f.	p-value
27 + 46 = …… (multiple choice)						
14 – 6 = ……. (matching)	No	109	3.830	0.537		
9 + 7 = ……. (matching)					224	0.985
8 x 7 = ……. (matching)	Yes	117	3.829	0.421		
PART B	Worksheet	N	Mean	S. D.	d.f.	p-value
17 + 20 = …….						
……. = 38 + 10						
13 + 51 = 51 + …….	No	109	4.817	2,241		
12 + ... = 28 + 4					224	
160 = ……. – 30						0.000
38 – 12 = ……. – 3						
……. = 17	Yes	117	6.017	2.228		
5 x 4 = ……. + 2						
PART C	Worksheet	N	Mean	S. D.	d.f.	p-value
Problem 1: Form an equation						
Problem 2: Form an equation	No	109	2.234	1.397	224	
Is 10 – 4 = 6 – 2 = 4 x 2 = 8 correct?						0.000
Is 3 + 8 = 11 x 6 = 66 – 5 = 61 correct?	Yes	117	3.308	2.274		

Finally the male and female mean scores for the experimental group were 13.15 and 13.16 respectively. For the control group both mean scores were 10.88. These mean scores indicate no differences in the performance of male and female pupils in the two groups.

4. Conclusions

The strength of earlier studies was their in-depth examination of single concepts: problems of interpretation, examinations of textbook content and influence of remedial intervention. Since there is now widespread agreement concerning these aspects, the researcher deemed it timely to draw on this previous knowledge and to combine his findings with these so as to increase the generalizability and in doing so to open the way to an innovative multi-faceted approach to the problem of misinterpretation. In thus doing he connects the problem with a possible cause and an essentially simple solution and shows that it is possible for a teacher, institution or educational body to diagnose a problem and to develop a solution in an economically timely manner. He suggests that this multi-faceted approach is both relevant and necessary for educational practitioners and researchers as well as curricula designers.

In the first phase of the study the existence of the problem of misinterpreting the relational meaning of the equal sign was investigated with the aid of a specifically designed test. Results showed that pupils in the sixth grades had no problem with simple arithmetic operations in which the equal sign was not involved (e.g. matching questions) or in standard operations-equals-answer form of equalities. However, the percentage of correct answers dropped by approximately 17% when the operations were shifted to the right hand side of the equality.

Difficulties in the interpretation of the equal sign were evident in the cases where equations involved operations-on-both-sides, as previously suggested by Falkner, Levi and Carpenter (1999). Even in operations with simple numbers, pupils in the final grade of primary school have substantial difficulty in interpreting the equal sign as a symbol of mathematical equivalence supporting findings by many researchers such as Behr et al. (1976), Kieran (1981) and Li et al. (2008). The findings of this study also concur with Rittle-Johnson, Mathews and Taylor (2011) and McNeil et al. (2006) among others, who argue that many children view the equal sign operationally, as a command to carry out arithmetic operations. This was also evident in the last two problems where pupils were asked to say whether the equation provided (leading to the solution) was correct. Even though the final answers were correct, both equations were mathematically incorrect, treating the equal sign operationally. In both cases none of the students responded with a NO (the option indicating that the equation was incorrect).

Seo and Ginsburg (2003) argue that children's pre-school experiences and most importantly their early elementary school instruction exclusively involves the standard operations-equals-answer forms of equations and this leads to the incomplete interpretation of the meaning of the equal sign. Li et al. (2008) give maths textbooks in China, which include multiple representations of equivalence and where pupils exhibit an excellent understanding of the relational meaning of the equal sign, as an example. They, like Powel and Fuchs (2010), McNeil and Alibali (2005a), Rittle-Johnson and Alibili (1999) and Wolters (1991), argue that the context in which teachers and textbooks present the equal sign plays a major role in the development of children's complete understanding of the equal sign. The review of the maths textbooks in the second phase of this study supports their argument in the Cypriot context and revealed some important findings. In the first four years of their elementary schooling, children in Cyprus come across the operations-on-both-sides form of equalities only 31 times and this amounts to just 1% of the total number of equalities in their textbooks. Notably in the first grade textbooks there are no instances of operations-on-both-sides forms. This context of equation has been shown by McNeil and Alibali (2005a) to elicit the relational interpretation of the equal sign. With regard to the textbooks for the final two grades of the primary school, 2.9% of the cases in the fifth grade and 6.3% in the sixth grade are in this specific form. Furthermore, the operations-on-the-RHS form of equations appears only 51 times in the textbooks of the first four grades and this amounts to just 1.7% of the cases. Such heavy focus on the standard, operations-equal-answer, equalities in textbooks and by extension in school work "leads students to an operational understanding of the equal sign" (Powel & Fuchs 2010, p. 383).

In the third phase of this study, children in an experimental group received instruction in the form of a worksheet specifically designed to promote the relational meaning of the equal sign. Their teachers, under the guidance of the researcher, spent two to three periods explaining and solving the worksheet exercises to their pupils. After a period of four weeks, the original test from phase 1 was administered to the experimental group and an equivalent sized control group of pupils (who had received no specific instruction on the equal sign) in three schools.

The worksheet-based instruction was not targeted at the test but at a more complete understanding of the relational meaning of the equal sign, and this is why the researcher asked teachers to let four weeks pass by before administering the test, without any additional preparation for it. The intention was to assess the impact of brief specific instruction, after a reasonable lapse of time, on the understanding of the equal sign.

The results of the test were highly significant. The children in the experimental group outperformed those in the control group, not only in their total test score, but also separately in the two parts of the test that contained non-standard equations and word problems.

McNeil et al. (2006) cite Piaget and colleagues who suggest that children's difficulties with the understanding of the relational meaning of the equal sign may be due to developmental cognitive limitations. The findings of this study absolve developmental cognitive limitations at least for the final two grades of the primary school, ages 10 to 12. Children who received just 2-3 periods of instruction on the relational meaning of the equal sign exhibited better understanding four weeks later in an unexpected test on this important concept. This suggests that introducing the relational meaning of the equal sign in the fifth and sixth grade, ages 10+ (and perhaps even earlier), with appropriate arithmetic operations for each age group will have a positive impact on the understanding of this vital mathematical concept. This also suggests that the pupils in the final two grades of the primary schools under investigation have most probably never been given instruction targeted at the proper relational meaning of the equal sign.

In Cypriot primary education, maths textbooks (and by extension the teachers) place emphasis on doing mathematical operations (addition, subtraction, multiplication and division) correctly rather than assisting children in interpreting the equal sign as a relational symbol of mathematical equivalence. Findings of the study substantiate Fielker's (2008) report that the concept of equality is ignored in the context of learning mathematics. Fielker suggests that inclusion in textbooks, in the curriculum and in class work of non-standard equalities and especially in the form operations-on both-sides will have a positive impact on children's relational understanding of the equal sign. This finding also brings out a weakness of centralized educational systems such as the Cypriot one. Textbooks in centralized systems are selected or prepared centrally with restricted flexibility for the teachers. Autonomy on the other hand motivates and encourages educators to think of students' needs and to devise suitable syllabi and more importantly in the context of this study, to design or choose and use the most appropriate textbooks for those needs.

The importance of this study lies in the completeness of its multi-faceted approach. The diagnosis of the problem was proceeded by an investigation, through a thorough exploration of the textbooks used by this age group (10 to 12 years of age) and followed by a statistical experiment to see if remedial action has a positive impact on the pupils' understanding of the equal sign.

The findings confirm those reported in various studies conducted elsewhere. This confirmation of previous findings in a Cypriot sample of pupils contributes to existing knowledge as it adds to their generalizability and points to the emphasis that should be placed on this important concept. "The equal sign needs at least as much attention as other common symbols we use in mathematics" (Fielker 2008). Teachers should be made aware of the possible misconceptions and the dangers of children having a narrow comprehension of the equal sign and give their pupils explicit instruction. Also, authors and curricula designers should ensure that the relational meaning of the symbol is explained and that a variety of equations in all forms are included in curricula as well as in primary school maths textbooks. After all children can only learn what they are taught and not what we assume they may have been taught.

4.1 Possible limitations of the study

This study has three possible limitations. First the sample comes from primary schools in Limassol and not from the entire population of Cypriot primary schools. This entails the danger that the results may not be representative of the whole Cypriot primary pupil population. Second, in the final phase of the study the researcher did not observe the administration of the worksheet by the teachers; instead he accepted their assurances that his guidelines were followed implicitly. Finally, this study is limited in that data for the effect of instruction specific to the relational meaning of the equal sign were collected from a single source, a single administration of a test. Asking follow up questions or conducting interviews could have given more information on children's understanding. However, this was beyond the scope of the already multistep structured study.

The researcher suggests further research in two different directions. First the impact calculators have on the misinterpretation of the equal sign, and more in-depth investigation through teacher and pupil interviews for the effect of specific instruction.

5. Acknowledgements

The researcher wishes to thank the following people:

Kyriaki Papademetriou, an ex-student of the researcher who helped in the data collection in phase 1 of this study.
Stelios Antoniades, a fellow maths teacher of the researcher who helped in the construction of the worksheet used in phase 3 and in reviewing some of the maths textbooks in phase 2.
Miranda Jane Walker, a language teacher, colleague of the researcher, for proof-reading the final version of this manuscript.

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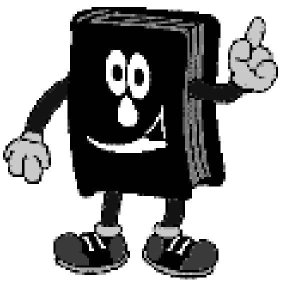
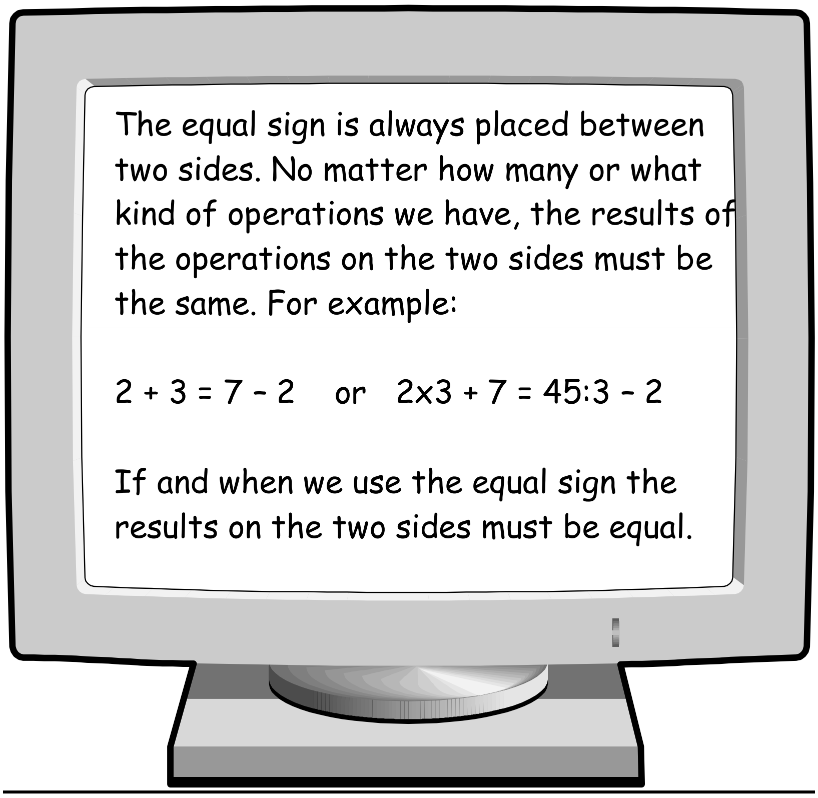
Appendix

<u>Worksheet – Interpreting the ‘ = ’ sign</u>

Today we will talk about the equal sign. Please read carefully what I have written on the screen of the computer on the right and copy it in your exercise book. Then do the exercises that follow.





Exercise 1

Match each operation in the first row with the appropriate answer in the second, as shown in the example.

12 + 4 3x10 + 2x3 180 ×7 6- 2 x 3

5 - 5 960 21 - 5 12x3 180×10-180×3

Exercise 2

Fill in the gaps with the numbers that make the following equalities hold.

(<u>Reminder</u>: The result of all the operations on the one side must be the same as the result of all operations on the other)

<u>Additions - subtractions</u>

(α) 12 + ………… = 17
(β) 25 - ………… = 21
(γ) 12 = 10 + 7 - …………
(δ) 9 = 2x3 + …………
(ε) 15 = …………
(στ) 12 + 7 = 15 + …………
(ζ) 25 + 3 = ………… + 7
(η) 7 + 16 = 16 + …………
(θ) ………… = 29
(ι) 31 + 12 = 48 - …………
(κ) 27 + 5 = ………… - 6
(λ) 18 – 12 = 28 - …………
(μ) 24 – 10 = ………… - 2
(ν) 5 + ………… + 3 = 13 + 7
(ξ) 17 + 10 + 4 = ………… - 5
(ο) 8 + 4 + 8 = 3 + …………
(π) 125 = ………… + 12 – 7
(ρ) 10 + 10 + 10 – ………… = 25 - 2

<u>All operations</u>

(α) 12 : 3 = …………
(β) 25 x 4 = …………
(γ) 45 = ………… x 3
(δ) 9 = 36 : …………
(ε) 15 = ………… : 4
(στ) 12 x 3 = ………… - 4
(ζ) 12 x 5 = 10 x …………
(η) ………… = 7 x 7 – 9
(θ) ………… x ………… = 35
(ι) 3x5 + 4 = ………… - 2
(κ) 27 : 3 = ………… - 3
(λ) 18 – 12 = ………… : 3
(μ) 2 x 3 x 5 = ………… x 5 x 2
(ν) 5 + 5 + 5 = ………… x 5
(ξ) 7 x 2 = ………… - 3
(ο) 8 + 4 + 8 = ………… + 3x5
(π) 125 = ………… + 9 x 5
(ρ) 5 x 8 - ………… = 48 : 6

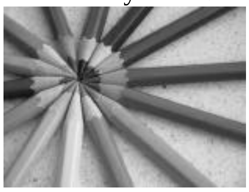
Exercise 3

Solve the following problems with the help of an equation. In the place of the unknown number use the letter x or ν or a box.

(<u>Reminder</u>: The result of all the operations on the one side must be the same as the result of all operations on the other)

Problem 1

Tasos has 8 pencils. Michael has 4 pencils more than Tasos. How many pencils do both boys have?



Problem 2

Irene and Stavros opened a candy box and started eating them. Irene ate 15 candies and Stavros ate twice the number that Irene ate. If there were 20 candies left in the box how many were there before the two children opened the box?

Exercise 3

Write a word problem that could be solved with the use of each of the following equations:

A) $10 + 5 - \nu = 12$ B) $(2 + 8) \times 5 = \nu$

Exercise 4

Below you are given a problem and a suggested solution which is mathematically incorrect. What is wrong with the solution? Write the correct solution.

(<u>Reminder</u>: The result of all the operations on the one side must be the same as the result of all operations on the other)

Problem: George, Maria and Peter went to the bookshop to buy books. George had 12 euros and Maria 15 euros. Peter had twice the amount of euros the other two children had together. How much money did Peter have?

Suggested solution: 15 + 12 = 27 x 2 = 54

Why is the solution mathematically incorrect?

……………………………………………………………………………………………

……………………………………………………………………………………………

Write the correct solution:

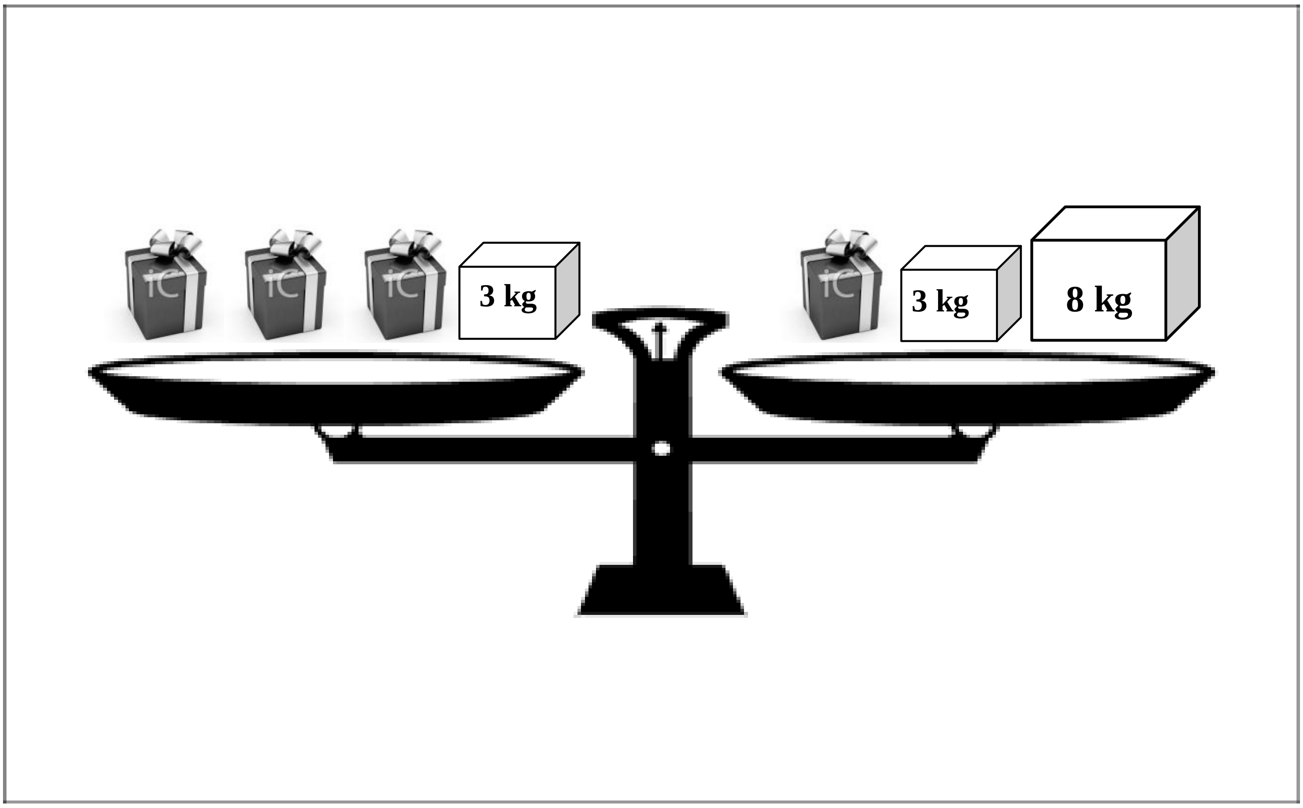
……………………………………………………………………………………………

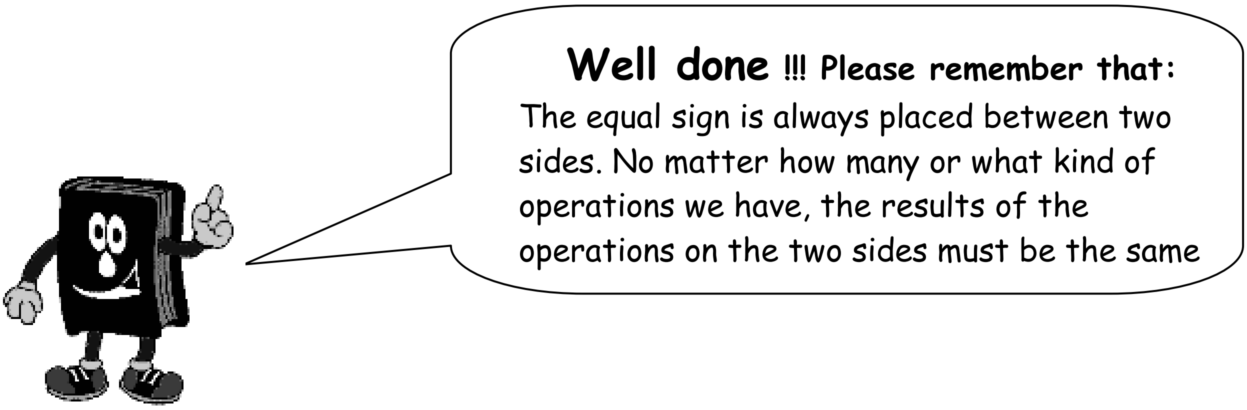
……………………………………………………………………………………………

Exercise 5

The picture on the next page shows scales in balance. On the scales there are four identical boxes of chocolates with exactly the same weight, two weights of
3 kg each and one weight of 8 kg. Describe the successive steps you would take in order to find the weight of one box of chocolates.

……
……
……
…………


3 kg
3 kg
8 kg


Well done !!! Please remember that:
The equal sign is always placed between two sides. No matter how many or what kind of operations we have, the results of the operations on the two sides must be the same